

# Eigenvalues of Laplacian with constant magnetic field on noncompact hyperbolic surfaces with finite area

Abderemane MORAME<sup>1</sup> and Françoise TRUC<sup>2</sup>

<sup>1</sup> *Université de Nantes, Faculté des Sciences, Dpt. Mathématiques, UMR 6629 du CNRS, B.P. 99208, 44322 Nantes Cedex 3, (FRANCE), E.Mail: morame@math.univ-nantes.fr*

<sup>2</sup> *Université de Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, B.P. 74, 38402 St Martin d'Hères Cedex, (France), E.Mail: Francoise.Truc@ujf-grenoble.fr*

## Abstract

We consider a magnetic Laplacian  $-\Delta_A = (id + A)^*(id + A)$  on a noncompact hyperbolic surface  $\mathbf{M}$  with finite area.  $A$  is a real one-form and the magnetic field  $dA$  is constant in each cusp. When the harmonic component of  $A$  satisfies some quantified condition, the spectrum of  $-\Delta_A$  is discrete. In this case we prove that the counting function of the eigenvalues of  $-\Delta_A$  satisfies the classical Weyl formula, even when  $dA = 0$ .<sup>1</sup>

## 1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface  $(\mathbf{M}, g)$  and a smooth, real one-form  $A$  on  $\mathbf{M}$ . We define the magnetic Laplacian, the Bochner Laplacian

$$-\Delta_A = (i d + A)^*(i d + A), \quad (1.1)$$

$$((i d + A)u = i du + uA, \forall u \in C_0^\infty(\mathbf{M}; \mathbb{C}).$$

The magnetic field is the exact two-form  $\rho_B = dA$ .

If  $dm$  is the Riemannian measure on  $\mathbf{M}$ , then

$$\rho_B = \tilde{\mathbf{b}} dm, \quad \text{with } \tilde{\mathbf{b}} \in C^\infty(\mathbf{M}; \mathbb{R}). \quad (1.2)$$

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The magnetic intensity is  $\mathbf{b} = |\tilde{\mathbf{b}}|$ .

It is well known, (see [Shu] ), that  $-\Delta_A$  has a unique self-adjoint extension on  $L^2(\mathbf{M})$ , containing in its domain  $C_0^\infty(\mathbf{M}; \mathbb{C})$ , the space of smooth and compactly supported functions. The spectrum of  $-\Delta_A$  is gauge invariant : for any  $f \in C^1(\mathbf{M}; \mathbb{R})$ ,  $-\Delta_A$  and  $-\Delta_{A+df}$  are unitarily equivalent, hence they have the same spectrum.

We are interested in constant magnetic fields on  $\mathbf{M}$  in the case when  $(\mathbf{M}, g)$  is a non-compact geometrically finite hyperbolic surface of finite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$\mathbf{M} = \bigcup_{j=0}^J M_j \quad (1.3)$$

where the  $M_j$  are open sets of  $\mathbf{M}$ , such that the closure of  $M_0$  is compact, and (when  $J \geq 1$ ) the other  $M_j$  are cuspidal ends of  $\mathbf{M}$ .

This means that, for any  $j$ ,  $1 \leq j \leq J$ , there exist strictly positive constants  $a_j$  and  $L_j$  such that  $M_j$  is isometric to  $\mathbb{S} \times ]a_j^2, +\infty[$ , equipped with the metric

$$ds_j^2 = y^{-2} ( L_j^2 d\theta^2 + dy^2 ) ; \quad (1.4)$$

( $\mathbb{S} = \mathbb{S}^1$  is the unit circle and  $M_j \cap M_k = \emptyset$  if  $j \neq k$ ).

Let us choose some  $z_0 \in M_0$  and let us define

$$d : \mathbf{M} \rightarrow \mathbb{R}_+ ; \quad d(z) = d_g(z, z_0) ; \quad (1.5)$$

$d_g(\cdot, \cdot)$  denotes the distance with respect to the metric  $g$ .

For any  $b \in \mathbb{R}^J$ , there exists a one-form  $A$ , such that the corresponding magnetic field  $dA$  satisfies

$$dA = \tilde{\mathbf{b}}(z) dm \quad \text{with} \quad \tilde{\mathbf{b}}(z) = b_j \quad \forall z \in M_j. \quad (1.6)$$

The following statement on the essential spectrum is proven in [Mo-Tr1] :

**Theorem 1.1** *Assume (1.3) and (1.6). Then for any  $j$ ,  $1 \leq j \leq J$  and for any  $z \in M_j$  there exists a unique closed curve through  $z$ ,  $\mathcal{C}_{j,z}$  in  $(M_j, g)$ , not contractible and with zero  $g$ -curvature. ( $\mathcal{C}_{j,z}$  is called an horocycle of  $M_j$ ). The following limit exists and is finite:*

$$[A]_{M_j} = \lim_{d(z) \rightarrow +\infty} \int_{\mathcal{C}_{j,z}} A. \quad (1.7)$$

If  $J^A = \{j \in \mathbb{N}, 1 \leq j \leq J \text{ s.t. } [A]_{M_j} \in 2\pi\mathbb{Z}\} \neq \emptyset$ , then

$$\text{sp}_{ess}(-\Delta_A) = [\frac{1}{4} + \min_{j \in J^A} b_j^2, +\infty[. \quad (1.8)$$

If  $J^A = \emptyset$ , then  $\text{sp}_{ess}(-\Delta_A) = \emptyset$  :  
 $-\Delta_A$  has purely discrete spectrum, (its resolvent is compact).

When the magnetic Laplacian  $-\Delta_A$  has purely discrete spectrum, it is called a magnetic bottle, (see [Col2]).

If  $A = df + A^H + A^\delta$  is the Hodge decomposition of  $A$  with  $A^H$  harmonic, ( $dA^H = 0$  and  $d^*A^H = 0$ ), then  $\forall j$ ,  $[A]_{M_j} = [A^H]_{M_j}$ , so the discreteness of the spectrum of  $-\Delta_A$  depends only on the harmonic component of  $A$ . So one can see the case  $J^A = \emptyset$  as an Aharonov-Bohm phenomenon [Ah-Bo], a situation where the magnetic field  $dA$  is not sufficient to describe  $-\Delta_A$  and the use of the magnetic potential  $A$  is essential : we can have magnetic bottle with null intensity.

## 2 The Weyl formula in the case of finite area with a non-integer class one-form

Here we are interested in the pure point part of the spectrum. We assume that  $J^A = \emptyset$ , then the spectrum of  $-\Delta_A$  is discrete. In this case, we denote by  $(\lambda_j)_j$  the increasing sequence of eigenvalues of  $-\Delta_A$ , (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda, -\Delta_A) = \sum_{\lambda_j < \lambda} 1. \quad (2.1)$$

We will show that the asymptotic behavior of  $N(\lambda)$  is given by the Weyl formula :

**Theorem 2.1** *Consider a geometrically finite hyperbolic surface  $(\mathbf{M}, g)$  of finite area, and assume (1.6) with  $J^A = \emptyset$ , (see (1.7 for the definition).*

*Then*

$$N(\lambda, -\Delta_A) = \lambda \frac{|\mathbf{M}|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda). \quad (2.2)$$

**Remark 2.2** As  $J^A$  depends only on the harmonic component of  $A$ ,  $J^A$  is not empty when  $\mathbf{M}$  is simply connected. In [Go-Mo] there are some results close to Theorem 2.1, but for simply connected manifolds.

The cases where the magnetic field prevails were studied in [Mo-Tr1] and in [Mo-Tr2].

**Proof of Theorem 2.1.** Any constant depending only on the  $b_j$  and on  $\min_{1 \leq j \leq J} \inf_{k \in \mathbb{Z}} |[A]_{M_j} - 2k\pi|$  will be denoted invariably  $C$ .

Consider a cusp  $M = M_j = \mathbb{S} \times ]\alpha^2, +\infty[$  equipped with the metric  $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$  for some  $\alpha > 0$  and  $L > 0$ .

Let us denote by  $-\Delta_A^M$  the Dirichlet operator on  $M$ , associated to  $-\Delta_A$ . The first step will be to prove that

$$N(\lambda, -\Delta_A^M) = \lambda \frac{|M|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda). \quad (2.3)$$

Since  $-\Delta_A^M$  and  $-\Delta_{A+d\varphi+kd\theta}^M$  are gauge equivalent for any  $\varphi \in C^\infty(\overline{\mathbf{M}}; \mathbb{R})$  and any  $k \in \mathbb{Z}$ , we can assume that

$$-\Delta_A^M = L^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}, \quad \text{with } A_1 = -\xi \pm b L e^{-t}, \quad \xi \in ]0, 1[,$$

( $b = b_j$ ,  $2\pi\xi - [A]_M \in 2\pi\mathbb{Z}$ ). Then we get that

$$\text{sp}(-\Delta_A^M) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell); \quad P_\ell = D_t^2 + \frac{1}{4} + \left( e^t \frac{(\ell + \xi)}{L} \pm b \right)^2,$$

for the Dirichlet condition on  $L^2(I; dt)$ ;  $I = ]\alpha^2, +\infty[$ . This implies that

$$N(\lambda, -\Delta_A^M) = \sum_{\ell \in \mathbb{Z}} N(\lambda, P_\ell) = \sum_{\ell \in X_\lambda} N(\lambda, P_\ell) \quad (2.4)$$

with  $X_\lambda = \{ \ell / e^{\alpha^2} \frac{|\ell + \xi|}{L} < \sqrt{\lambda - 1/4} - b \}$ .

Denoting by  $Q_\ell$  the Dirichlet operator on  $I$  associated to

$$Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell + \xi)^2}{L^2} e^{2t},$$

we easily get that

$$Q_\ell - C\sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C\sqrt{Q_\ell}. \quad (2.5)$$

Therefore one can find a constant  $C(b)$ , depending only on  $b$ , such that, for any  $\lambda \gg 1 + C(b)$ ,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \leq N(\lambda, P_\ell) \leq N(\lambda + \sqrt{\lambda}C(b), Q_\ell). \quad (2.6)$$

Following Titchmarsh's method ([Tit], Theorem 7.4) we establish the following bounds

**Lemma 2.3** *There exists  $C > 1$  so that for any  $\mu \gg 1$  and any  $\ell \in X_\mu$ ,*

$$w_\ell(\mu) - \pi \leq \pi N(\mu - \frac{1}{4}, Q_\ell) \leq w_\ell(\mu) + \frac{1}{12} \ln \mu + C, \quad (2.7)$$

with

$$\begin{aligned} w_\ell(\mu) &= \int_{\alpha^2}^{+\infty} \left[ \mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dt \\ &= \int_{\alpha^2}^{T_{\mu,L}} \left[ \mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dt; \end{aligned} \quad (2.8)$$

$$(e^{T_{\mu,L}} = L\sqrt{\mu}/(\inf_{k \in \mathbb{Z}} |\xi - k|)) .$$

### Proof of Lemma 2.3

The lower bound is easily obtained (see [Tit], Formula 7.1.2 p 143) so we focus on the upper bound.

Let us define  $V_\ell = \frac{(\ell + \xi)^2}{L^2} e^{2t}$  and denote by  $\phi_\mu^\ell$  a solution of  $Q_\ell \phi = (\mu - \frac{1}{4})\phi$ . Consider  $x_\ell$  and  $y_\ell$  so that  $V_\ell(x_\ell) = \mu$  and  $V_\ell(y_\ell) = \nu$ , for a given  $0 < \nu < \mu$  to be determined later. We denote by  $m$  the number of zeros of  $\phi_\mu^\ell$  on  $] \alpha^2, y_\ell[$ . Recall that the number  $n$  of zeros of  $\phi_\mu^\ell$  on  $] \alpha^2, x_\ell[$  is equal to  $N(\mu - \frac{1}{4}, Q_\ell)$ . Applying Lemma 7.3 p 146 in [Tit] we deduce that

$$m\pi = \int_{\alpha^2}^{y_\ell} [\mu - V_\ell]^{1/2} dt + R_\ell$$

with  $R_\ell = \frac{1}{4} \ln(\mu - V_\ell(\alpha^2)) - \frac{1}{4} \ln(\mu - V_\ell(y_\ell)) + \pi$ , hence

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2} dt| \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2} + R_\ell + (n - m)\pi$$

According to the Sturm comparison theorem ([Tit], p 107-108), we have

$$(n - m)\pi \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2}$$

and

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2} dt| \leq \ln\left(\frac{\mu}{\nu}\right)(\mu - \nu)^{1/2} + \frac{1}{4} \ln \mu - \frac{1}{4} \ln(\mu - \nu) + 2\pi$$

Now taking  $\nu = \mu - \mu^{2/3}$  we get the desired estimate.

In view of (2.4) we now compute  $\sum_{\ell \in \mathbb{Z}} w_\ell(\mu)$ . We first get the following

**Lemma 2.4** *There exists  $C > 1$  such that, for any  $\mu \gg 1$  and any  $t \in [\alpha^2, T_{\mu,L}]$ ,*

$$\left| \int_{\mathbb{R}} \left[ \mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dx - \sum_{\ell \in \mathbb{Z}} \left[ \mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} \right| \leq C(\sqrt{\mu} + \frac{e^t}{L}).$$

This leads to

**Lemma 2.5** *There exists  $C > 1$  such that, for any  $\mu \gg 1$ ,*

$$\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[ \mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dx dt - \sum_{\ell \in \mathbb{Z}} w_\ell(\mu) \right| \leq C\sqrt{\mu} \ln \mu.$$

We now compute the integral in the left-hand side.

Making the change of variables  $y^2 = \frac{(x+\xi)^2}{L^2\mu} e^{2t}$  we obtain that it is equal to  $\mu L \int_{\alpha^2}^{T_{\mu,L}} e^{-t} dt \int_{\mathbb{R}} [1 - x^2]_+^{1/2} dx$ , so we get

**Lemma 2.6** *There exists  $C > 1$  such that, for any  $\mu \gg 1$ ,*

$$\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[ \mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dx dt - \mu L e^{-\alpha^2} \int_{\mathbb{R}} [1 - x^2]_+^{1/2} dx \right| \leq C\sqrt{\mu}.$$

Noticing that  $|M| = 2\pi L e^{-\alpha^2}$  and using Lemmas 2.5 and 2.6 we have

**Lemma 2.7**

$$\frac{1}{\pi} \sum_{\ell} \in w_\ell(\mu) = \frac{|M|}{4\pi} \mu + \mathbf{O}(\sqrt{\mu} \ln \mu), \quad \text{as } \mu \rightarrow +\infty.$$

In view of (2.4),(2.6) and (2.7) Lemma 2.7 ends the proof of formula (2.3).

Now it remains to consider the whole surface  $\mathbf{M}$ .

We have :  $\mathbf{M} = \left( \bigcup_{j=0}^J M_j \right)$

where the  $M_j$  are open sets of  $\mathbf{M}$ , such that the closure of  $M_0$  is compact, and the other  $M_j$  are cuspidal ends of  $\mathbf{M}$  and

$M_j \cap M_k = \emptyset$ , if  $j \neq k$ . We denote  $M_0^0 = \mathbf{M} \setminus \left( \bigcup_{j=1}^J \overline{M_j} \right)$ , then

$$\mathbf{M} = \overline{M_0^0} \cup \left( \bigcup_{j=1}^J \overline{M_j} \right). \quad (2.9)$$

Let us denote respectively by  $-\Delta_{A,D}^\Omega$  and by  $-\Delta_{A,N}^\Omega$  the Dirichlet operator and the Neumann-like operator on an open set  $\Omega$  of  $\mathbf{M}$  associated to  $-\Delta_A$ . The minimax principle and (2.9) imply that

$$\begin{aligned} N(\lambda, -\Delta_{A,D}^{M_0^0}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_{A,D}^{M_j}) &\leq N(\lambda, -\Delta_A) \\ &\leq N(\lambda, -\Delta_{A,N}^{M_0^0}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_{A,N}^{M_j}) \end{aligned} \quad (2.10)$$

The Weyl formula with remainder, (see [Hor] for Dirichlet boundary condition and [Sa-Va] p. 9 for Neumann-like boundary condition), gives that

$$\left\{ \begin{array}{l} N(\lambda, -\Delta_{A,D}^{M_0^0}) = (4\pi)^{-1} |M_0^0| \lambda + \mathbf{O}(\sqrt{\lambda}) \\ N(\lambda, -\Delta_{A,N}^{M_0^0}) = (4\pi)^{-1} |M_0^0| \lambda + \mathbf{O}(\sqrt{\lambda}) \end{array} \right\}. \quad (2.11)$$

The asymptotic formula for  $N(\lambda, -\Delta_{A,N}^{M_j})$ ,

$$N(\lambda, -\Delta_{A,N}^{M_j}) = \lambda \frac{|M_j|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda), \quad (2.12)$$

is obtained as for the Dirichlet case (2.3) (with  $M = M_j$ ), by noticing that  $N(\lambda, P_{\ell,D}) \leq N(\lambda, P_{\ell,N}) \leq N(\lambda, P_{\ell,D}) + 1$ , where  $P_{\ell,D}$  and  $P_{\ell,N}$  are Dirichlet and Neumann operators on a half-line  $I = ]\alpha^2, +\infty[$ , associated to the same differential Schödinger operator  $P_\ell = D_t^2 + \frac{1}{4} + (e^{t \frac{(\ell + \xi)}{L}} \pm b)^2$ .

We get (2.2) from (2.3) with  $M = M_j$ , (2.12), (for any  $j = 1, \dots, J$ ), (2.10) and (2.11).  $\square$

**Remark 2.8** *Theorem 2.1 still holds if the metric of  $\mathbf{M}$  is modified in a compact set.*

*When  $A = 0$ ,  $-\Delta = -\Delta_0$  has embedded eigenvalues in its essential spectrum,  $(sp_{ess}(-\Delta) = [\frac{1}{4}, +\infty[)$ . If  $N_{ess}(\lambda, -\Delta)$  denotes the number of these eigenvalues in  $[\frac{1}{4}, \lambda[$ , then it is well known that one has an upper bound  $N_{ess}(\lambda, -\Delta) \leq \lambda \frac{|\mathbf{M}|}{4\pi}$ ; see [Col1] and [Hej] for the history and related improvement of the upper bound. Recently [Mul] established a sharp asymptotic formula, similar to our case,*

$$N_{ess}(\lambda, -\Delta) = \lambda \frac{|\mathbf{M}|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda),$$

*for some particular  $\mathbf{M}$ .*

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